Problem 1.36

(a) Show that

$$\int_{\mathcal{S}} f(\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \int_{\mathcal{S}} [\mathbf{A} \times (\nabla f)] \cdot d\mathbf{a} + \oint_{\mathcal{P}} f\mathbf{A} \cdot d\mathbf{l}. \tag{1.60}$$

(b) Show that

$$\int_{\mathcal{V}} \mathbf{B} \cdot (\nabla \times \mathbf{A})\, d\tau = \int_{\mathcal{V}} \mathbf{A} \cdot (\nabla \times \mathbf{B})\, d\tau + \oint_{\mathcal{S}} (\mathbf{A} \times \mathbf{B}) \cdot d\mathbf{a}. \tag{1.61}$$

Solution

Part (a)

Use Identity 7 inside the front cover to rewrite the integrand and then use Stokes's theorem to turn the latter surface integral into a closed loop integral.

$$\begin{aligned}
\int_{\mathcal{S}} f(\nabla \times \mathbf{A}) \cdot d\mathbf{a} &= \int_{\mathcal{S}} [\mathbf{A} \times (\nabla f) + \nabla \times (f\mathbf{A})] \cdot d\mathbf{a} \\
&= \int_{\mathcal{S}} [\mathbf{A} \times (\nabla f)] \cdot d\mathbf{a} + \int_{\mathcal{S}} [\nabla \times (f\mathbf{A})] \cdot d\mathbf{a} \\
&= \int_{\mathcal{S}} [\mathbf{A} \times (\nabla f)] \cdot d\mathbf{a} + \oint_{\mathcal{P}} f\mathbf{A} \cdot d\mathbf{l}
\end{aligned}$$

Part (b)

Use Identity 6 inside the front cover to rewrite the integrand and then use Gauss's theorem to turn the latter volume integral into a closed surface integral.

$$\begin{aligned}
\int_{\mathcal{V}} \mathbf{B} \cdot (\nabla \times \mathbf{A})\, d\tau &= \int_{\mathcal{V}} [\mathbf{A} \cdot (\nabla \times \mathbf{B}) + \nabla \cdot (\mathbf{A} \times \mathbf{B})]\, d\tau \\
&= \int_{\mathcal{V}} \mathbf{A} \cdot (\nabla \times \mathbf{B})\, d\tau + \int_{\mathcal{V}} \nabla \cdot (\mathbf{A} \times \mathbf{B})\, d\tau \\
&= \int_{\mathcal{V}} \mathbf{A} \cdot (\nabla \times \mathbf{B})\, d\tau + \oint_{\mathcal{S}} (\mathbf{A} \times \mathbf{B}) \cdot d\mathbf{a}
\end{aligned}$$